Algebraic Model Counting

Roderick de Britto Heemskerk

Luka Janjić

TU Delft

WMC reminder

- Theory *T* over variables *V*
- Weight function $w : L \rightarrow \mathbb{R}^+$
- Set of models *M(T)*

$$\mathbf{WMC}(T) = \sum_{I \in \mathcal{M}(T)} \prod_{l \in I} w(l)$$

AMC as a generalization of WMC

$$\mathbf{WMC}(T) = \sum_{I \in \mathcal{M}(T)} \prod_{l \in I} w(l)$$

$$\downarrow$$

$$\mathbf{A}(T) = \bigoplus_{I \in \mathcal{M}(T)} \bigotimes_{l \in I} \alpha(l)$$

Commutative Semiring

- Set A equipped with:
  - $\bigoplus \rightarrow$ generalized addition
  - $\bigotimes \rightarrow$ generalized multiplication
  - $e^{\oplus} \in A \rightarrow$ additive identity
  - $e^{\otimes} \in A \rightarrow$ multiplicative identity

Commutative Semiring – Laws

Associativity of $\oplus$ and $\otimes$:

- $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

Commutativity of $\oplus$ and $\otimes$:

- $a \oplus b = b \oplus a$

Distributivity of $\otimes$ over $\oplus$:

- $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$

Annihilation by $e^{\oplus}$:

- $e^{\oplus} \otimes a = e^{\oplus}$

Example semiring: $(\mathbb{N}, +, \cdot, 0, 1)$

- Addition and multiplication are associative, commutative and distributive
- $X + 0 = X$
- $X \cdot 1 = X$
- $X \cdot 0 = 0$
- $(\mathbb{N}, +, \cdot, 0, 1)$ is a commutative semiring

Examples

- WMC $\rightarrow$ $(\mathbb{R}_{\geq 0}, 0, 1, +, \cdot)$, $\alpha(l) \in \mathbb{R}_{\geq 0}$
- PROB $\rightarrow$ $(\mathbb{R}_{\geq 0}, 0, 1, +, \cdot)$, $\alpha(l) \in [0,1]$
- S-PATH $\rightarrow$ $(\mathbb{N}^{\infty}, \infty, 0, \min, +)$, $\alpha(l) \in \mathbb{N}$
- GRAD $\rightarrow$ *recall the guest lecture*
- many more…

NNF

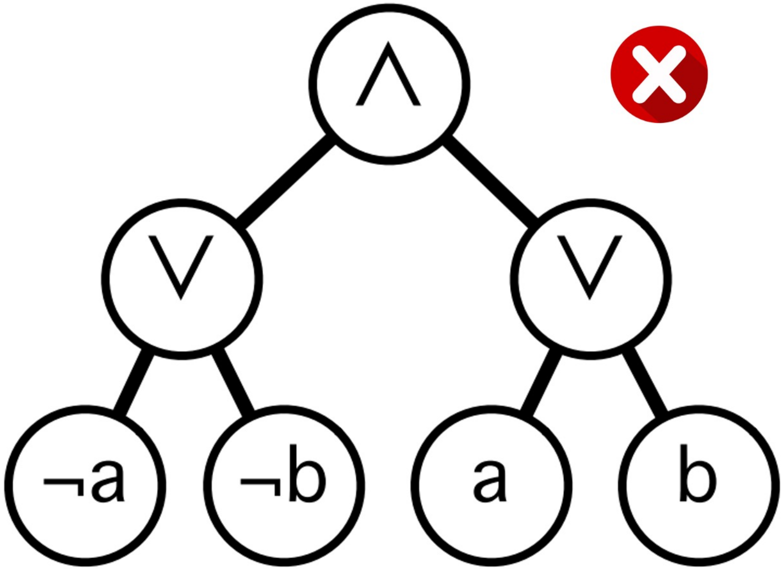
- Negation Normal Form
- Class of formulae
- Directed Acyclic Graph

Properties

- Decomposable
  - Children of a conjunction node share no variables.
- Deterministic
  - Only one of the children of disjunction nodes can be True at a time.
- Smooth
  - All children of disjunction nodes have the same variables.

Decomposability

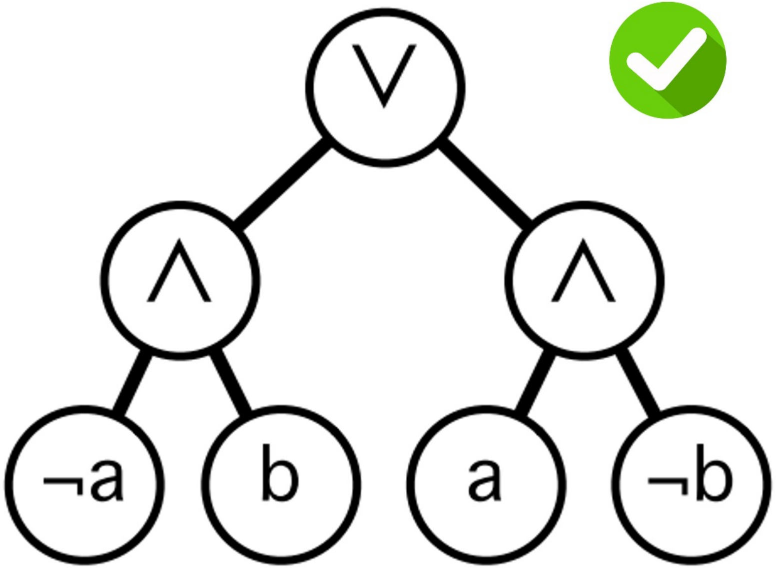
Children of a conjunction node share no variables.



Decomposability

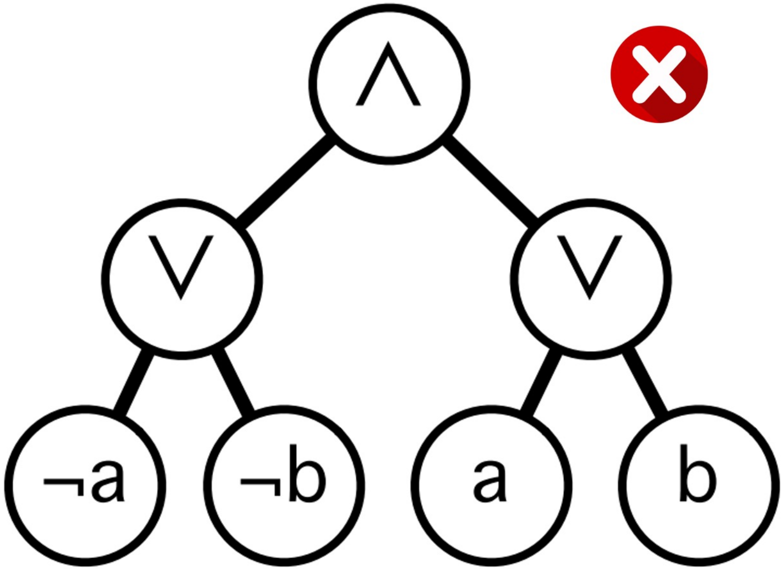
Children of a conjunction node share no variables.

DNNF



Determinism

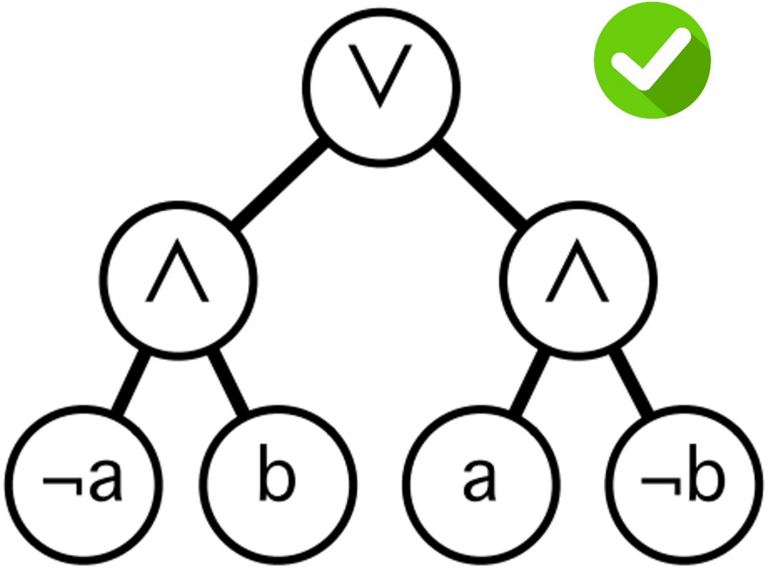
Only one of the children of disjunction nodes can be True at a time.



Determinism

Only one of the children of disjunction nodes can be True at a time.

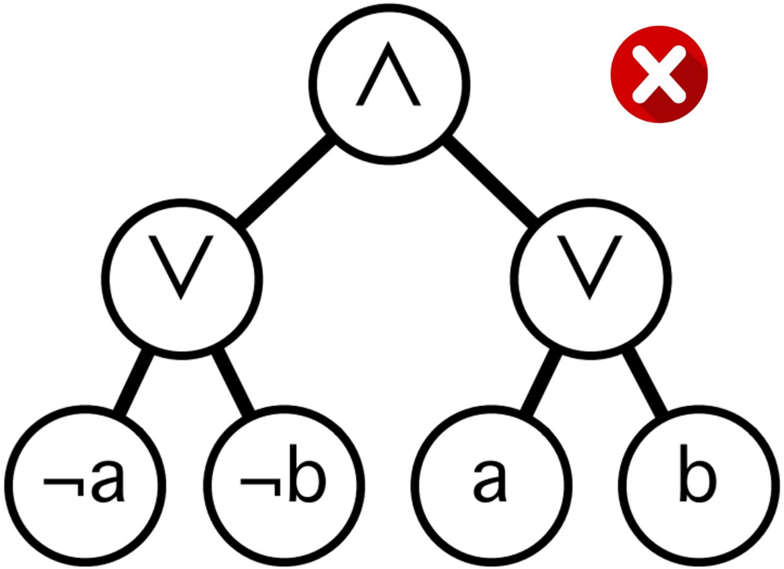
d-NNF



TU Delft

Smooth

All children of disjunction nodes have the same variables.

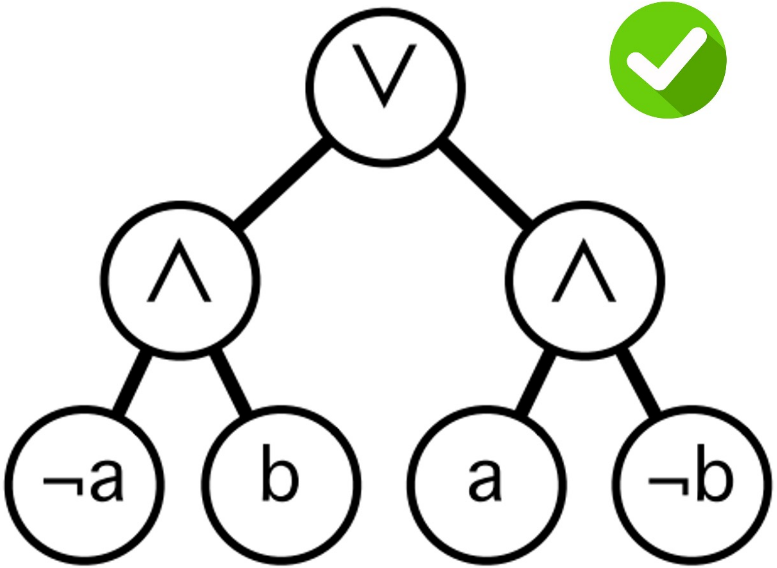


TU Delft

Smooth

All children of disjunction nodes have the same variables.

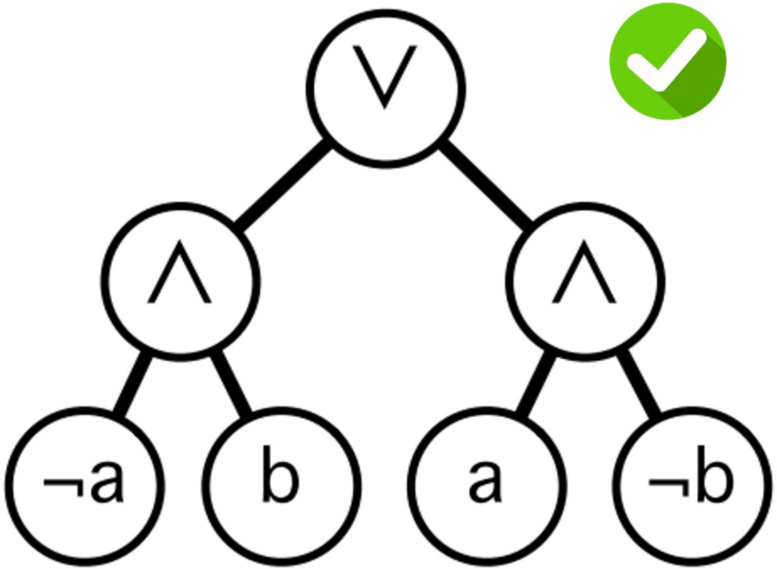
s-NNF



Smooth

All children of disjunction nodes have the same variables.

sd-DNNF



Succinctness

- Definition
- Compactness
- More succinct → Less space and Quicker execution

Let’s put it together!

What do we have so far

- Problems as **sums-of-products** over a commutative semiring
- Compile them to compact form (**NNF**s) for easy computation
- Three properties: **smooth, deterministic, decomposable**
- **Succinctness** as a measure of how "compact" can we make it

Naive solution – MODS

- Represent *all models M(T)* explicitly
- MODS ⊂ sd-DNNF
- **Exponentially less succinct** than the rest

Expression in sd-DNNF:

$f(x,y) = x$

x	y	f(x,y)
0	0	0
0	1	0
1	0	1
1	1	1

Equivalent MODS:

$(x \land \neg y) \lor (x \land y)$

Can we make it better?

Yes, but...

- **s**mooth, **d**eterministic, **D**ecomposable NNFs
  - More succinct than MODS
  - Still contains MODS
- Removing requirements on the NNF:
  - Increases succinctness
  - Imposes requirements on the semiring
  - Fewer problems fit in

Evaluation

function $\textsc{Eval}(N, \oplus, \otimes, e^{\oplus}, e^{\otimes}, \alpha)$
  if N is a true node $\top$ **then return** $e^{\otimes}$
  if N is a false node $\bot$ **then return** $e^{\oplus}$
  if N is a literal node l **then return** $\alpha(l)$
  if N is a disjunction $\bigvee_{i=1}^{m} N_i$ **then**
    return $\bigoplus_{i=1}^{m} \textsc{Eval}(N_i, \oplus, \otimes, e^{\oplus}, e^{\otimes}, \alpha)$
  if N is a conjunction $\bigwedge_{i=1}^{m} N_i$ **then**
    return $\bigotimes_{i=1}^{m} \textsc{Eval}(N_i, \oplus, \otimes, e^{\oplus}, e^{\otimes}, \alpha)$

Evaluation

Base cases:

if N is a true node $\top$ **then return** $e^{\otimes}$
if N is a false node $\bot$ **then return** $e^{\oplus}$
if N is a literal node l **then return** $\alpha(l)$

Recursive cases:

if N is a disjunction $\bigvee_{i=1}^{m} N_i$ **then**
  return $\bigoplus_{i=1}^{m} \text{EVAL}(N_i, \oplus, \otimes, e^{\oplus}, e^{\otimes}, \alpha)$
if N is a conjunction $\bigwedge_{i=1}^{m} N_i$ **then**
  return $\bigotimes_{i=1}^{m} \text{EVAL}(N_i, \oplus, \otimes, e^{\oplus}, e^{\otimes}, \alpha)$

AMC in a nutshell

- A general framework fitting many problems
- **Commutative semiring** $(A, \oplus, \otimes, 0, 1)$ to model the problem
- Labeling function $\alpha : L \rightarrow A$ to evaluate literals
- What **properties** does it satisfy? – The more the better
  - Idempotent $\oplus \rightarrow$ relax determinism
  - Neutral $(\oplus, \alpha) \rightarrow$ relax smoothness
  - Consistency preserving and idempotent $(\otimes, \alpha)$

    $\rightarrow$ relax decomposability

TU Delft